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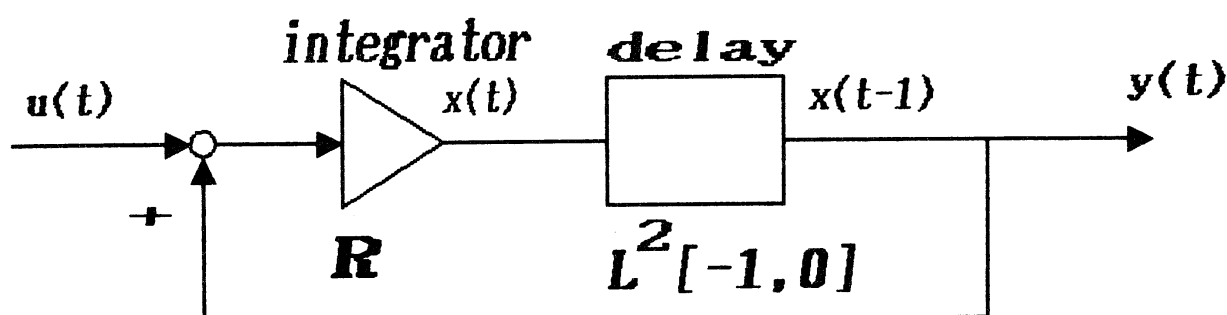
線形ダイナミカルシステムのモデル, 微分作用素, 可制御性

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1. Introduction

Consider the following delay-differential system:

$$\begin{aligned} \dot{x}(t) &= x(t-1) + u(t), & u(t) &: \text{input vector} \\ y(t) &= x(t-1). & y(t) &: \text{output vector.} \end{aligned} \quad (1)$$



Block diagram for the above system

Clearly, we need to have a function space on  $[0, 1]$  (or  $[-1, 0]$ ) to store the last one second behavior for the state-space model. (Hale [6] and others.)

A well-known standard choice is:

$$X = \mathbb{R} \times L^2[-1, 0] \quad (\text{called an } M_2 \text{ space})$$

by Delfour, Mitter, and others ([3,4]). It induces the following functional differential equation:

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x_t \\ z_t \end{bmatrix} &= \begin{bmatrix} z_t(-1) \\ (\partial/\partial \theta) z_t(\theta) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), & z_t(0) &= x_t \\ y &= z_t(-1). \end{aligned} \quad (2)$$

This model has been effectively used for many purposes, say, optimal control, feedback stabilization, etc. Recently, there is even a control

scheme by actively using a delay element in the compensator (called repetitive control: [10], [18]).

Question: Where does this function space (and the model (2)) come from?

## 2. Spectrum, Eigenfunction Completeness and Reachability.

Standard Realization Procedure ([15, 16, 17]):

Basic Idea: Use the left shift  $\sigma_t$  in  $L^2_{1,\infty}[0,\infty)$  as a universal model.

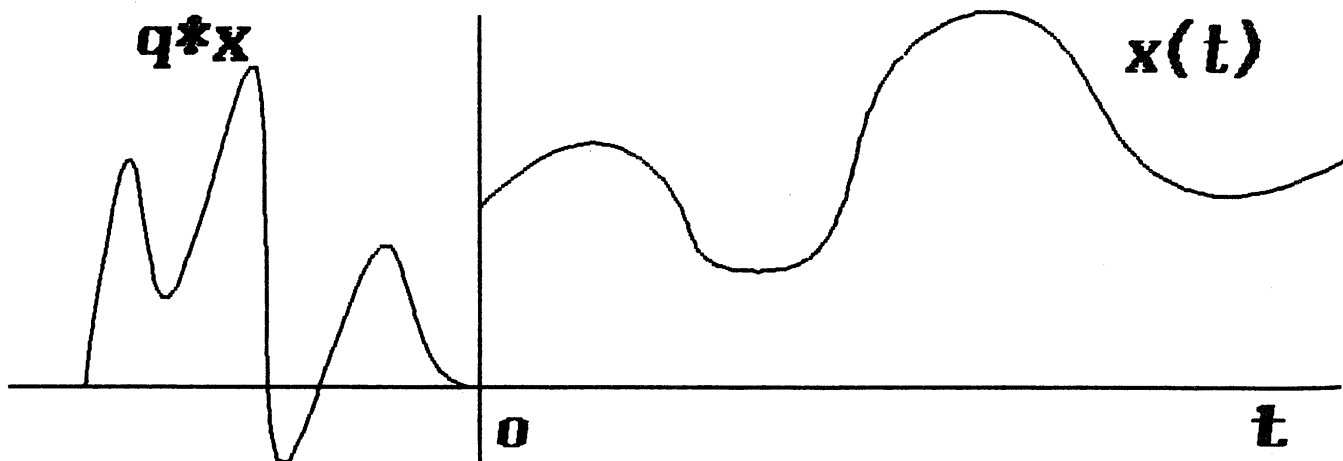
1) Express the input/output relationship of system (1) as  $y = A*u$  where  $A$  is the impulse response of (1).

2) Express  $A$  as the ratio  $q^{-1}*p$  of distributions with compact support in  $(-\infty, 0]$ . In (1),  $A = \delta / \{\delta_{-1}' - \delta\}$ . (If this is possible,  $A$  is called pseudo-rational.)

3) Take the closed subspace

$$X^q := \{x \in L^2_{1,\infty}[0,\infty); \text{supp } (q*x) \text{ in } (-\infty, 0]\} \quad (3)$$

as the state space and  $\sigma_t$  in  $X^q$  as the generating semigroup for state transition.



In the above example,  $X^q$  is given by the closure, taken in  $L^2_{loc}[0, \infty)$ , of the space of solutions of the equation

$$(d/dt)x(t+1) = x(t), \text{ for } t \geq 0.$$

It is readily seen that this space is isomorphic to

$$\mathbb{R} \times L^2[0, -1].$$

4) The desired functional differential equation model is then given by

$$(d/dt)x_t(\cdot) = Fx_t(\cdot) + A(\cdot)u(t) \quad (4)$$

where  $F$  : infinitesimal generator of  $\sigma_t$ .

Questions on the above construction:

- a) What is the meaning of  $\text{supp}(q*x)$  in  $(-\infty, 0]$ ?
- b) When does  $q$  have compact support in  $(-\infty, 0]$ ?
- c) What is  $F$ ?
- d) What is  $\sigma(F)$ ?
- e) What is the space  $M$  of (generalized) eigenvectors of  $F$ ?
- f) When is  $M$  dense in  $X^q$ ?
- h) When is system (4) reachable?

Some Remarks and Answers:

On a), b): Paley-Wiener Theorem:

Theorem (Paley-Wiener-Schwartz [14])  $q$  is a distribution with compact support contained in  $(-\infty, 0]$  iff

$\hat{q}(s)$  is an entire function of  $s$  such that

$$|\hat{q}(s)| \leq C(1+|s|)^m \exp(a \cdot \text{Re } s), \quad \text{Re } s \geq 0$$

$$\leq C(1+|s|)^m, \quad \operatorname{Re} s \leq 0 \quad (5)$$

This implies

$$x \in X^q \Leftrightarrow \operatorname{supp} (q*x) \text{ in } (-\infty, 0] \quad (6)$$

(Note  $\operatorname{supp} (q*x)$  is always compact.)

$\Rightarrow$  all singularities of  $\hat{x}(s)$  are cancelled by  $\hat{q}(s)$ .

On c):  $F = d/dt$  (better to write  $d/d\tau$  by change of variable). **Remark:**  
The model (2) is actually obtained by the above realization procedure. Somewhat surprisingly, the right-hand side operator in (2) is actually the differential operator  $d/d\tau$  represented in the space  $\mathbb{R} \times L^2[0,1]$ , which is isomorphic to  $X^q$ . For details, see [17].

On d): Spectrum of  $F$ .

Let us compute the point spectrum only.

$$(\lambda I - F)x = 0 \Leftrightarrow dx/dt = \lambda x, \quad x \in X^q$$

$$\Leftrightarrow H(t)\exp(\lambda t) \in X^q$$

$$\Leftrightarrow \hat{q}(s) \cdot 1/(s-\lambda) \text{ satisfies the Paley-Wiener estimate (5).}$$

$$\Leftrightarrow \hat{q}(s) \cdot 1/(s-\lambda) \text{ is an entire function.}$$

$$\Leftrightarrow \hat{q}(\lambda) = 0.$$

Actually, we can prove that ([15])

i) if  $\hat{q}(\lambda) \neq 0$  then  $\lambda \in \rho(F)$ .

Therefore,

ii) every  $\lambda \in \sigma(F)$  is an eigenvalue (with finite multiplicity).

On e): Let  $m :=$  order of  $\lambda$  as a zero of  $\hat{q}(s)$ .

Then the generalized eigenspace  $M_\lambda$  corresponding to  $\lambda$  is

$\text{span} \{ \exp(\lambda t), t \exp(\lambda t), \dots, t^m \exp(\lambda t) \}.$

$\Rightarrow M = \text{span}_{\lambda, m} \{ \exp(\lambda t), t \exp(\lambda t), \dots, t^m \exp(\lambda t) \}.$

$\lambda, m$

On f):  $M$  is dense in  $X^q$

$$\Leftrightarrow x^* \in (X^q)', \langle x^*, x \rangle = 0 \text{ for all } x \in M \Rightarrow x^* = 0 \quad (7)$$

[REMARK] This question is closely related to the question of reachability, feedback stabilization, etc., and has been studied via the state space representation as in (2) by a number of authors: [7], [8], [9], [10], [12], [13], etc. (some of them only study reachability). However, a concrete algebraic criterion is fairly difficult to obtain, and has been obtained via somewhat ad hoc methods for delay-differential systems (e.g., [9], [11], [13]). We here attempt to pursue a more unified and systematic approach for pseudo-rational systems, which are known to include the class of delay-differential systems.

Our question is then: What is  $(X^q)'$ ?

$$\begin{aligned} [\text{LEMMA 1}] \quad (X^q)' &\simeq \bigcup L^2[-n, 0] / q^*(\bigcup L^2[-n, 0]) \\ &= \varinjlim L^2[-n, 0] / q^*(\varinjlim L^2[-n, 0]) \end{aligned} \quad (8)$$

Proof. Omitted. A standard fact from locally convex duality, and the fact that  $L^2_{1, \infty}[0, \infty)$  is the projective limit of  $\{L^2[0, n]\}$ .  $\square$

$$\begin{aligned} [\text{LEMMA 2}] \quad \langle x^*, x \rangle = 0 \text{ for all } x \in M &\Leftrightarrow \\ x^{\wedge}(s) / q^{\wedge}(s) &= \text{entire function of } s. \end{aligned}$$

Proof. For simplicity, assume  $q^{\wedge}(s)$  has simple roots only. The duality

in Lemma 1 is ([16])

$$\langle \varphi, x \rangle := \int \varphi(t)x(-t)dt = \int \varphi(-t)x(t)dt,$$

$$\varphi \in \lim L^2[-n, 0], \quad x \in X^q.$$

Then  $\langle x^*, x \rangle = 0$  for all  $x \in M \Leftrightarrow$

$$\langle x^*, \exp(\lambda t) \rangle = x^*(\lambda) = 0, \quad \text{any } \lambda \text{ such that } q^*(\lambda) = 0.$$

$\Leftrightarrow x^*(s)/q^*(s)$  is entire.  $\square$

Therefore, we have proved

$$x^* \perp M \Leftrightarrow x^*(s) = q^*(s)\varphi(s) \text{ for some entire function } \varphi(s). \quad (8)$$

If any such  $\varphi$  were the Laplace transform of a distribution with compact support in  $(-\infty, 0]$ , then  $M$  would be dense in  $X^q$ , i.e. this system is eigenfunction complete.

Let us first prove that  $\varphi$  is always the Laplace transform of a distribution with compact support not necessarily contained in  $(-\infty, 0]$ .

To this end, we need to prove, in view of the Paley-Wiener theorem (5), that

i)  $\varphi(s)$  is an entire function of exponential type;

ii) it has polynomial growth on the imaginary axis.

We give a proof for i) only (for details, see [19]).

Proof of i) By the well-known Hadamard factorization theorem ([2]) for entire functions, it is clear that  $\varphi$  is of order 1, i.e.,

for any  $\varepsilon > 0$ , there exists  $R > 0$  such that

$$|\varphi(s)| < \exp(|s|^{1+\varepsilon})$$

for  $|s| > R$ .

We must quote the following deep result by Lindelöf from complex analysis:

[Lindelöf's theorem] ([2]) Let  $f$  be an entire function of order 1. Let  $\lambda_1, \dots, \lambda_n, \dots$  be the zeros of  $f(s)$ , counted according to multiplicity.

Define

$$n(r) := \text{no. of zeros of } f \text{ in } |s| < r$$

$$S(r) := \sum_{|\lambda_n| \leq r} 1/\lambda_n$$

Then  $f(s)$  is of exponential type, i.e.,  $|f(s)| \leq C \exp(K|s|)$  iff

- i)  $n(r) = O(r)$ ;
- ii)  $S(r)$  is bounded.

Proof of  $\varphi = \text{exponential type}$ .

Let  $\{\lambda_1, \dots, \lambda_n, \dots\}$  be the zeros of  $q^\wedge(s)$ , and  $\{\mu_1, \dots, \mu_n, \dots\}$  the zeros of  $\varphi(s)$ . Then the zeros of  $x^\wedge(s) = \{\lambda_1, \dots, \lambda_n, \dots\} \cup \{\mu_1, \dots, \mu_n, \dots\}$ .

- i)  $n_\varphi(r) = O(r)$  is obvious since  $x^\wedge(s)$  satisfies this property

$$\begin{aligned} \text{ii) } |S_\varphi(r)| &= |S_{x^\wedge}(r) - S_{q^\wedge}(r)| \\ &\leq |S_{x^\wedge}(r)| + |S_{q^\wedge}(r)|, \end{aligned}$$

so that  $S_\varphi(r)$  is also bounded.  $\square$

Suppose now that we have agreed that  $\varphi$  is indeed the Laplace transform of a distribution with compact support. (To show this we need a little more work to ensure that  $\varphi(s)$  is of polynomial growth on the imaginary axis; see [19].) In view of the fact (8),

eigenfunction completeness  $\Leftrightarrow \text{supp } \varphi \subset (-\infty, 0]$  for all such  $\varphi$ .

(9)



Question: When is  $\text{supp } \varphi \subset (-\infty, 0]$ ?

Define

$$r(\varphi) := \sup \{t; t \in \text{supp } \varphi\}.$$

[LEMMA 3] Suppose  $r(\varphi), r(\psi) < \infty$ . Then

$$r(\varphi * \psi) = r(\varphi) + r(\psi).$$

Indication of Proof.

$$r(\varphi * \psi) \leq r(\varphi) + r(\psi) \text{ is obvious.}$$

To prove the reverse inequality, we need to show

$\varphi, \psi$  do not vanish in a neighborhood of endpoints  $a, b$

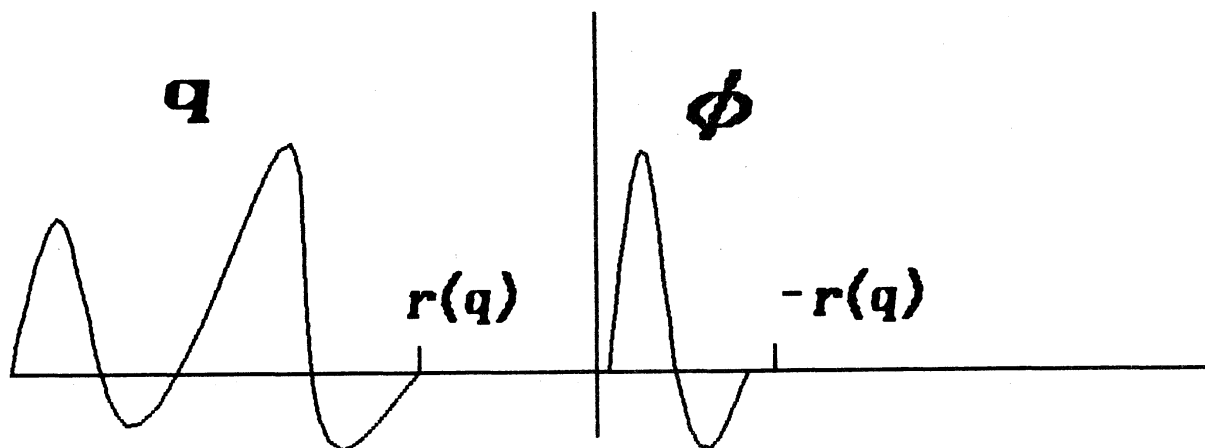
$$\Rightarrow \varphi * \psi \text{ does not vanish in a neighborhood of } a+b.$$

This follows from the local version of the Titchmarsh convolution theorem ([5]). (Need to go back to the original proof, or a proof by Miksinski; the usual proof ([20]) covers only the global version.)  $\square$

[THEOREM 1] The system (4) is eigenfunction complete (i.e.,  $M$  is dense in  $X^q$ ) iff  $r(q) = 0$ .

Proof. Observe that  $r(\varphi * q) = r(x^*) \leq 0$  and  $r(\varphi * q) = r(\varphi) + r(q)$ .

If  $r(q)=0$  then  $r(\varphi)=r(x^*)\leq 0$ .



Conversely, if  $r(q) < 0$ , then any  $\varphi$  with  $\text{supp } \varphi \subset (0, -r(q))$  gives rise to an  $x^*$  such that  $\varphi := x^* \cdot q^{-1}$

has the property

i)  $\varphi^*(s)$  is entire, and  $r(\varphi) > 0$ .

This contradicts statement (9), whence the eigenfunction completeness.

□

Let us now consider the reachability (controllability) question.

[DEFINITION] The system (4) is said to be quasi-reachable if the set of all elements in  $X^q$  that can be driven from 0 by a suitable application of an input is dense in  $X^q$ . It is said to be spectrally reachable if any element in  $M$  is reachable from 0 by an action of an input.

[LEMMA 4] The above system is spectrally reachable iff

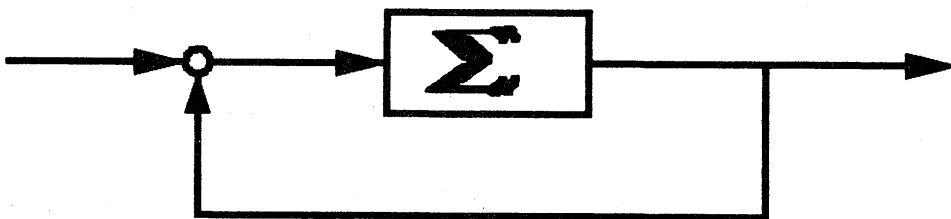
$$\text{rank } [q^*(\lambda) \ ; \ p^*(\lambda)] = \text{full for any } \lambda \in \mathbb{C}.$$

In the present scalar case, this is equivalent to:

no common zero between  $q^*(s)$  and  $p^*(s)$ .

Proof. Omitted. □

[LEMMA 5] Let  $\Sigma$  be a system.  $\Sigma$  is quasi-reachable iff the following system is quasi-reachable.



Combining the above lemmas together, we have

[THEOREM 2] The system (4) defined via  $X^q$  is quasi-reachable iff

- i)  $\text{rank } [q^*(\lambda) \mid p^*(\lambda)] = \text{full for any } \lambda \in \mathbb{C}; \text{ and}$
- ii)  $\max \{r(q), r(p)\} = 0.$

Sketch of Proof. We only prove the sufficiency. For details, see [19].

Case I)  $r(q) = 0$ . In this case, the space  $M$  of eigenfunctions is already dense. Since by i) the system is spectrally reachable, i.e., every element in  $M$  is reachable, we must have quasi-reachability.

Case II)  $r(q) < 0$  but  $r(p) = 0$ . In this case, form the feedback system in the above diagram. Then the new system has the impulse response  $(q+p)^{-1}p$ , i.e., we have a new denominator  $(q+p)$ . Clearly,  $r(q+p) = 0$ . Then by Lemma 5 and the above argument in Case I), the result follows.  $\square$

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